Math 3236 Statistical Theory

1/19/23

Exponential dist par λ.

$f(t) = \lambda e^{-\lambda t}$ $t \geq 0$.

waiting Time

From the results of a sample we want to estimate λ.

T_i $i = 1 \dots N$

T_i are i.i.d

They all have the same distribution $f(t)$ and they are independent.

$E(T_i) = \frac{1}{\lambda}$

$= \int_0^{\infty} t \lambda e^{-\lambda t} \, dt =$

$\lambda t = x$

$$= \frac{1}{\lambda} \int_0^\infty x e^{-x} \, dx$$

$$\mathbb{E}(T^2) = \int_0^\infty t^2 \lambda e^{-\lambda t} \, dt =$$

$x = \lambda t$

$$= \frac{1}{\lambda^2} \int_0^\infty x^2 e^{-x} \, dx$$

Ex.: $\int_0^\infty x^2 e^{-x} \, dx =$

$$e^{-x} = -\frac{d}{dx} e^{-x}$$

$$= -x^2 e^{-x} \Big|_0^\infty + 2\int_0^\infty x e^{-x} \, dx =$$

$= 2$

$$\mathbb{E}(T^2) = \frac{2}{\lambda^2}$$

$$V(T) = \mathbb{E}(T^2) - \mathbb{E}(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} =$$

$$= \frac{1}{\lambda^2}$$

$$\mathbb{E}(T_i) = \frac{1}{\lambda} \qquad V(T_i) = \frac{1}{\lambda^2}$$

T_i

$$\overline{T}_N = \frac{1}{N} \sum_i T_i \xrightarrow{P} \mathbb{E}(T) = \frac{1}{\lambda}$$

$$1 / \overline{T}_N \xrightarrow{P} \lambda$$

$\hat{\lambda}_N = 1 / \overline{T}_N$ is an estimator for λ.

$\hat{\lambda}_N$ is a r.v. and

$$\hat{\lambda}_N \to \lambda$$

$\hat{\lambda} = \frac{N}{\sum_{i=1}^{N} T_i}$ realization of my estimator

estimate of λ.

is $\hat{\lambda}$ unbiased ?

$\mathbb{E}(\hat{\lambda}) \overset{?}{=} \lambda$

What is the distribution of $\bar{T}$.

If T_1 and T_2 are exp with the same λ what is the p.d.f. of $T_1 + T_2$

$\lambda^2 t e^{-\lambda t} = f_{T_1+T_2}(t)$

$$S_N = \sum_{i=1}^{n} T_i \qquad T_i \text{ are exp ind}$$

$$f_{S_N}(t) = \frac{\lambda^N t^{N-1}}{(N-1)!} e^{-\lambda t}$$

$$\int_0^\infty \frac{\lambda^N t^{N-1}}{(N-1)!} e^{-\lambda t} dt \overset{?}{=} 1$$

$N=1 \qquad \lambda e^{-\lambda t}$

$N=2 \qquad \lambda^2 t e^{-\lambda t}$

$$x = \lambda t$$

$$\int_0^\infty \frac{\lambda^N t^{N-1}}{(N-1)!} e^{-\lambda t} dt = \int_0^\infty \frac{x^{N-1}}{(N-1)!} e^{-x} dx$$

$$\int_0^\infty x^{N-1} e^{-x} dx = \Gamma(N)$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$\Gamma(\alpha)$ Gamma function.

$$\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1)$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx =$$

$$= (\alpha - 1)\int_0^\infty x^{\alpha-2} e^{-x}\,dx$$

$$= (\alpha - 1)\,\Gamma(\alpha - 1)$$

$$\int_0^\infty x^{N-1} e^{-x}\,dx = \Gamma(N) =$$

$$(N-1)\,\Gamma(N-1) = (N-1)(N-2)\,\Gamma(N-2)$$

$$= (N-1)!$$

$$\int \frac{x^{N-1}}{(N-1)!}\, e^{-x}\,dx = 1$$

$$\int_0^{\infty} \frac{\lambda^N t^{N-1}}{(N-1)!} e^{-\lambda t} dt = 1$$

S_N has p.d.f.

$$\frac{\lambda^N t^{N-1}}{(N-1)} e^{-\lambda t}$$

$\mathbb{E}(S_N)$

$$S_N = \sum_{i=1}^{N} T_i$$

$$\mathbb{E}(S_N) = \sum_{i=1}^{N} \mathbb{E}(T_i) = N\,\mathbb{E}(T_e) =$$

$$= N/\lambda$$

$$\mathbb{E}\left(N/S_N\right) = \mathbb{E}(\hat{\lambda}) =$$

$$= N \int \frac{1}{t} \frac{\lambda^N t^{N-1}}{(N-1)!} e^{-\lambda t} dt =$$

$$= N \int \frac{\lambda^N t^{N-2}}{(N-1)!} e^{-\lambda t} \, dt =$$

$$= \frac{N\lambda}{N-1} \underbrace{\int \frac{\lambda^{N-1} t^{N-2}}{(N-2)!} e^{-\lambda t} \, dt}_{= 1}$$

Remember

$$\int \lambda^N \frac{t^{N-1}}{(N-1)!} e^{-\lambda t} \, dt = 1 \qquad \forall N$$

$$\mathbb{E}(\hat{\lambda}) = \frac{N}{N-1} \lambda = \left(1 + \frac{1}{N-1}\right) \lambda$$

$$\hat{\hat{\lambda}} = N-1 / S_N$$

$$\mathbb{E}(\hat{\hat{\lambda}}) = \lambda$$

Since $\frac{N-1}{N} \to 1$

$$\hat{\lambda} \xrightarrow{P} \lambda \, .$$

λ

$$\mathbb{E}(T_i) = m(\lambda)$$

$$\bar{T} = m(\lambda)$$

$$\hat{\lambda} = m^{-1}(\bar{T})$$

$\lambda \quad \theta$

$$\bar{T}_N \simeq \mathcal{N}\left(\frac{1}{\lambda}, \frac{1}{\lambda^2 N}\right)$$

$$\bar{T}_N = \frac{1}{N} \sum_i T_i$$

$$\frac{1}{\bar{T}_N}$$

$$\underbrace{\lambda \sqrt{N}\left(\bar{T}_N - \frac{1}{\lambda}\right)} \Rightarrow \mathcal{N}(0,1)$$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{T_i - \frac{1}{\lambda}}{1/\lambda}$$

$\overline{T}_N$ is almost always close to $\frac{1}{\lambda}$

$\frac{1}{\overline{T}_N}$ is almost always close to λ

$$\alpha(\overline{T}_N) = \alpha\left(\frac{1}{\lambda}\right) + \alpha'\left(\frac{1}{\lambda}\right)\left(\overline{T}_N - \frac{1}{\lambda}\right)$$

$$\frac{\lambda\sqrt{N}}{\alpha'(\lambda)}\left(\alpha(\overline{T}_N) - \alpha\left(\frac{1}{\lambda}\right)\right) = \sqrt{N}\left(\overline{T}_N - \frac{1}{\lambda}\right)\lambda$$

$$\frac{\lambda\sqrt{N}}{\alpha'(1/\lambda)}\left(\alpha(\overline{T}_N) - \alpha\left(\frac{1}{\lambda}\right)\right) \to N(0,1)$$